Chapter 1. Linear Equations in Linear Algebra

Section 1.1 Systems of Linear Equations

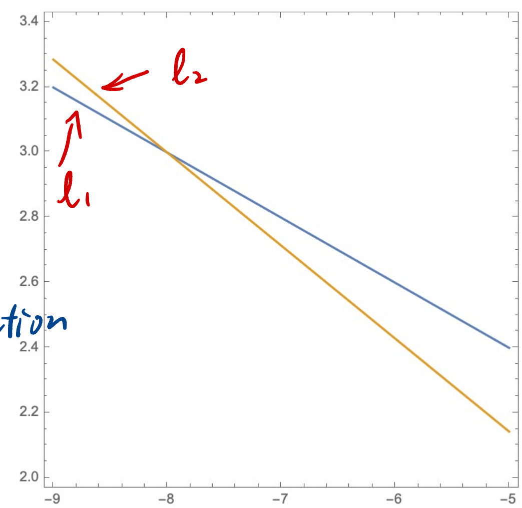
Example 0

$$\begin{aligned} x_1 + 5x_2 &= 7 \quad (\ell_1) \\ -2x_1 - 7x_2 &= -5 \quad (\ell_2) \end{aligned} \tag{1}$$

Remark:

1. It's easy to check $\begin{cases} x_1 = -8 \\ x_2 = 3 \end{cases}$ is the solution to Eq (1)

2. Geometrically, the graphs of the given two eqns are lines ℓ_1 and ℓ_2.
The solution to the system is the intersection point $(-8, 3)$ of ℓ_1 and ℓ_2



Basic Definitions

1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer.

For example, the two equations in Eq (1) are linear eqns.

Non-examples: $ax^2 + bx + c = 0$ (quadratic eqn) $\quad 3x^2 + 6y^2 = 18$

2. A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables-say, $x_1, \ldots, x_n$. An example is Eq (1).

3. A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For example,

$\begin{cases} x_1 = -8 \\ x_2 = 3 \end{cases}$ is a solution to system (1).

$\begin{cases} x_1 = 2 - x_2 \\ x_2 \text{ is any real number} \end{cases}$ is a solution to system $\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 4 \end{cases}$

4. The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. For example,

$$\begin{cases} x_1 + x_2 = 4 \\ x_1 - x_2 = -2 \end{cases} \quad \text{and} \quad \begin{cases} 2x_1 + x_2 = 5 \\ x_1 + 2x_2 = 7 \end{cases} \quad \text{are } \underline{\text{equivalent}}$$

since they have the same solution $\begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$

5. A system of linear equations has

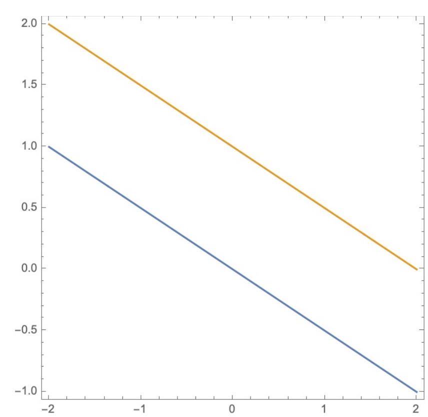
   (1) no solution, or

   (2) exactly one solution, or

   (3) infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution. For example,

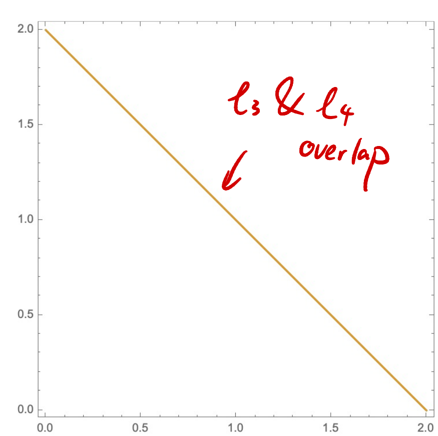
(1) $\begin{cases} x_1 + 2x_2 = 0 \\ x_1 + 2x_2 = 2 \end{cases}$ has no solution since otherwise we have $0 = 2$ (impossible!) So the system is inconsistent.



(2) Eq (1) in Example 0 has exactly one solution.

(3) $\begin{cases} x_1 + x_2 = 2 \; (\ell_3) \\ 2x_1 + 2x_2 = 4 \; (\ell_4) \end{cases}$ has infinitely many solutions.

So both systems (2) & (3) above are consistent.



Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a matrix. The **size** of a matrix tells how many rows and columns it has.

Example 1

$$
\begin{aligned}
x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\
x_2 - 7x_3 + 2x_4 &= -4 \\
5x_3 - x_4 &= 7 \\
x_3 + 3x_4 &= -5
\end{aligned}
\tag{2}
$$

The **coefficient matrix** of (2) is

since x_1 does not appear in the other 3 eqns

$$
\begin{bmatrix} 1 & 4 & -2 & 8 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}
$$

\# rows \# columns

It is a $\underline{4} \times \underline{4}$ (read "4 by 4") matrix

The **augmented matrix** of (2) is

$$
\left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \\ 0 & 0 & 1 & 3 & -5 \end{array}\right]
$$

\# rows \# columns

A $\underline{4} \times \underline{5}$ matrix

i.e., an augmented matrix of a system consists of the coefficient matrix with an added column containing the constant from the respective right sides of the eqns.

Solving a Linear System

Basic strategy: replace one system with an equivalent system that is easier to solve.

Let's go back to **Example 0**:

Example 1 Solve the system in **Example 0** by using elementary row operations on the equations or on the augmented matrix.

essentially the same process.

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

ANS: On the equations | On the augmented matrix

$$\begin{cases} x_1 + 5x_2 = 7 & \text{Eq ①} \\ -2x_1 - 7x_2 = -5 & \text{Eq ②} \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right]$$

$\Downarrow$ Eq ① $\times 2$ added Eq ② | $\Downarrow$ R1 $\times 2$ added to R2

$$\begin{cases} x_1 + 5x_2 = 7 & \text{Eq ①} \\ 0 + 3x_2 = 9 & \text{Eq ③} \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right]$$

$\Downarrow$ Eq ③ $/3$ | $\Downarrow$ R2 $/3$

$$\begin{cases} x_1 + 5x_2 = 7 & \text{Eq ①} \\ x_2 = 3 & \text{Eq ④} \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right]$$

$\Downarrow$ $-5 \times$ Eq ④ added to Eq ① | $\Downarrow$ $-5 \times$ R2 added R 1

$$\begin{cases} x_1 = -8 \\ x_2 = 3 \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right]$$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Remarks:

1. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.
2. If the augmented matrices of two linear systems are *row equivalent*, then the two systems have the *same solution set*.

Thus the matrices in Example 1 are row equivalent

The systems in Example 1 have the same solution set.

Existence and Uniqueness Questions

Two Fundamental questions about a linear system:

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution unique?

Example 2 The augmented matrix of a linear system has been reduced by row operations to the form shown. Continue the appropriate row operations and describe the solution set of the original system.

$$\begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array} \left[\begin{array}{ccc|c} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

ANS: Ordinarily, the next step would be interchange R3 and R4, to put a 1 in the third row and third column (we will talk more about this in § 1.2)

But R3 implies

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$, which means $0 = 1$ (impossible!)

So the system has no solutions.

Example 3 Do the three lines $x_1 - 4x_2 = 1, 2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

ANS: We consider the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right] \xrightarrow[\text{R1 added to R3}]{(-2)\times\text{R1 added to R2}} \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right]$$

$$\xrightarrow{\text{R2 added to R3}} \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right] \text{ which implies } \begin{cases} x_1 - 4x_2 = 1 \\ 7x_2 = -5 \\ 0 = 0 \end{cases}$$

The system is consistent and has only one solution.

So the 3 lines have only one point in common

The following two questions are left as exercises. I will provide the complete notes for solving them after the lecture. You will find similar questions in your homework assignments.

Exercise 4 Construct three different augmented matrices for linear systems whose solution set is $x_1 = 1, x_2 = 2, x_3 = -3$.

ANS: A basic principle of this section is that **row operations do not affect the solution set of a linear system**. Begin with a simple augmented matrix for which the solution is obviously $(1, 2, -3)$, and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set given by $(1, 2, -3)$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

Apparently, the first augmented matrix has a solution set given by $(1, 2, -3)$. The second augmented matrix is obtained from the first one by adding the second row with the first row. And the third augmented matrix is obtained from the second one by adding the third row with 2 times the first row.

Exercise 5 Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

ANS: It is not hard to see that $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$ by adding the second row with -2 times the first row.

Then the second equation is $(4 + 2h)x_2 = 0$ has a solution ($x_2 = 0$) for every value of h.

So the system is consistent for all h.